TILLING WITH ENVELOPES

REPRESENTING

What is a tile

It's about making patterns that fit perfectly together like a puzzle to fill up the plane. And in the era of digital cutting, this pattern is not necessarily a polygon ...

An historical result: With regular polygons, there are only three possible tilings: by squares, hexagons and triangles

There are many generalizations: periodic, a-periodic and quasiperiodic tilings, tilings of space.

Making a shape that will tile : the envelope method

From the viewpoint of history, this method was only recently discovered. It shows that in an area as well known (and as useful) as that of tilings, there still remains much to discover .... and to marvel at ...

The envelope method in 4 steps

1

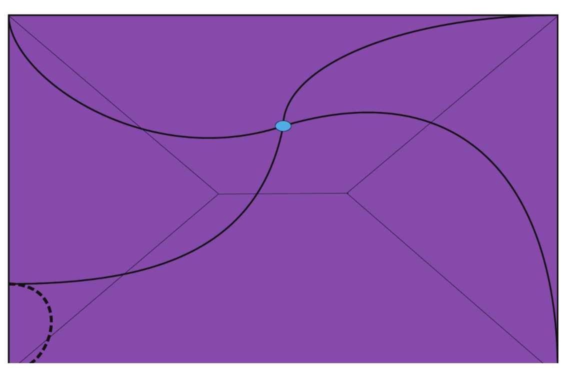
Choose two superimposable pieces of paper having the specific form of: an equilateral triangle, a square or a rectangle, a right angled isosceles triangle or a half square, a half equilateral triangle.

2

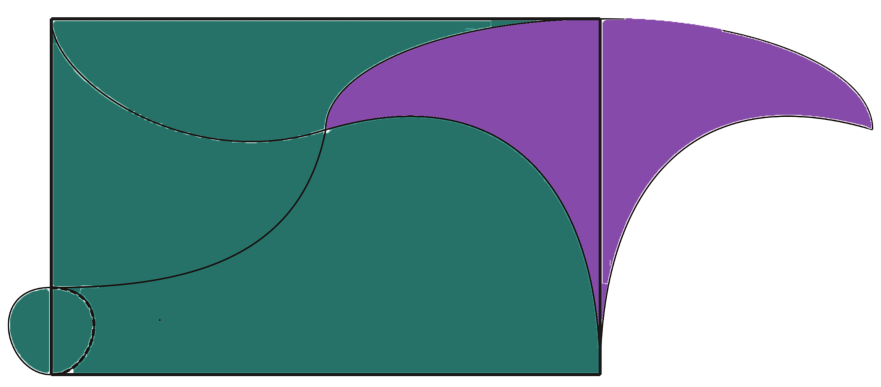
Tape these two pieces together to form a closed envelope

3

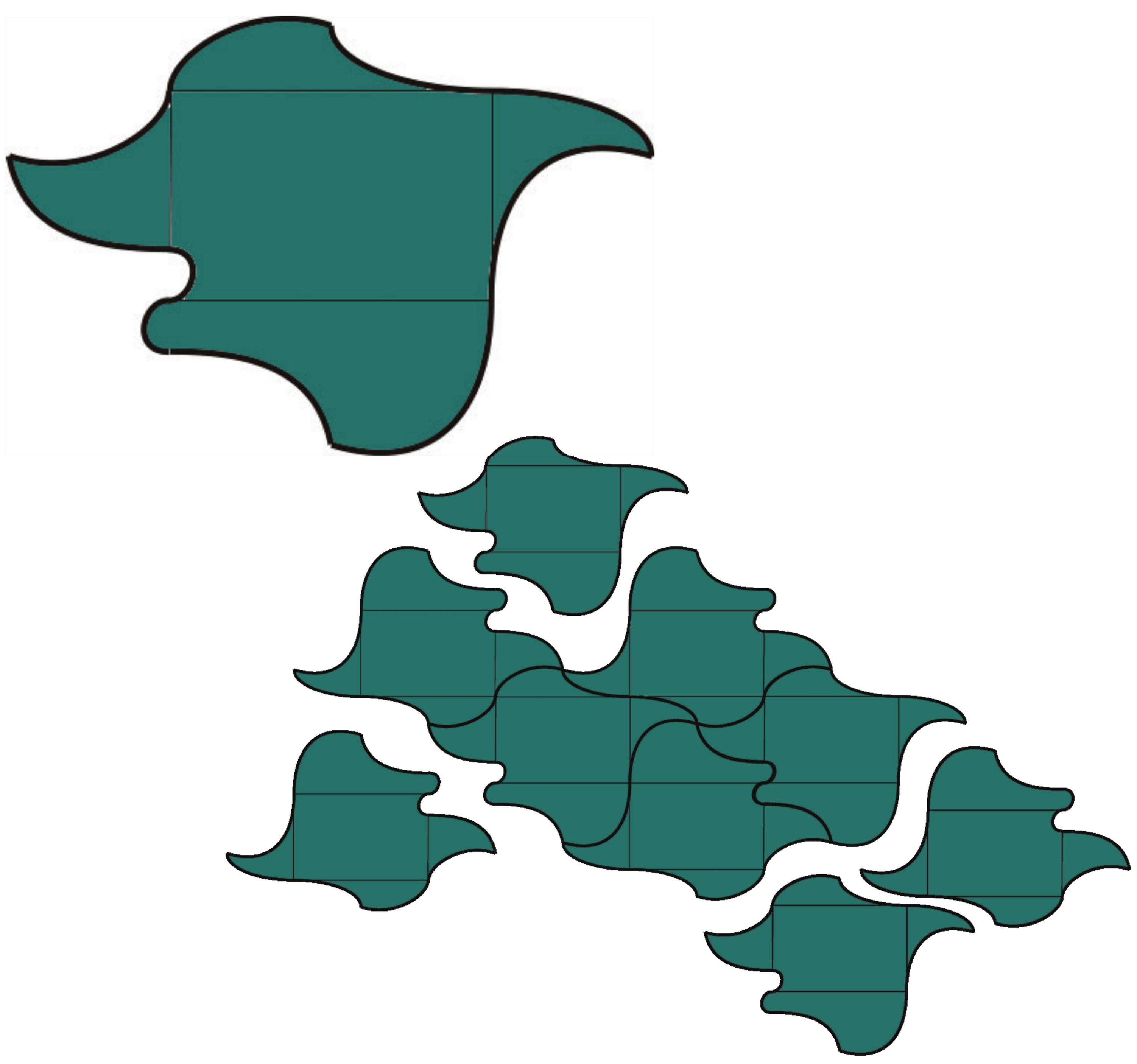
Choose one (or two) point (s) anywhere on one of the faces and draw curves connecting it to each of the angles of the envelope, passing in front of or behind the envelope. Warning: the lines should not be cut. Then cut according to the curves plotted on a single thickness of paper.



On unfolding 4

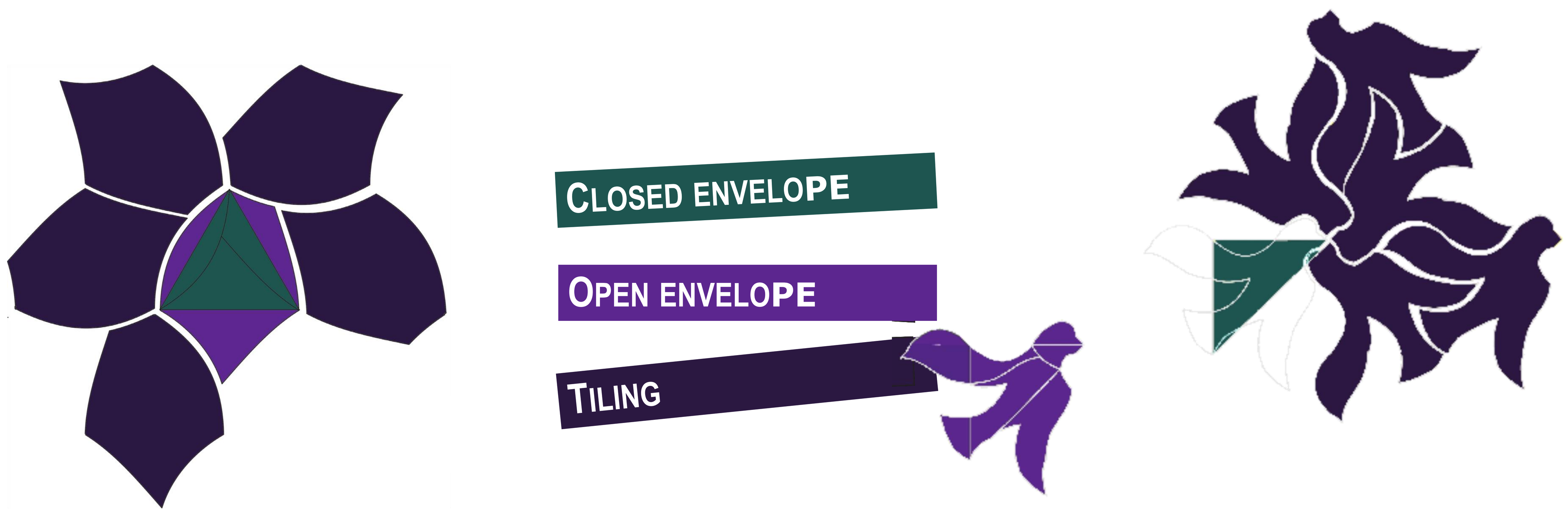


We obtain a tile, then a tiling



What if we imagine ...

A triangular envelope, what triangle should we choose?



To find all possible forms of envelopes?

Note that the rotations that leave the tiling unchanged have the vertices of the envelope as centers.

We also see that in the course of the tiling, the angle of rotation of the pattern about a vertex A of the envelope will be equal to twice the angle at the vertex A of the envelope. If we call "a" the angle at the vertex of the envelope, then the angle of rotation will be 2a.

Now, after a certain number of rotations through an angle 2a, we need to return exactly to the starting point, so that we have made a full turn of 360 °.

Thus 2a must divide 360° and so be of the form 360/n, with n an integer, which means that "a" is of the form 180 / n.

If the envelope is triangular, then the sum of the three angles must be 180°.

If we look for a quadrilateral envelope, then the sum of the 4 angles at the vertices must be 360°.

If we look for an envelope of pentagonal form, then the sum of the 5 angles at the vertices must be 540°, etc.

All the triangular envelopes ?

Hence, for a triangle, we are led to look for 3 integers *n, m, p* satisfying the relationship
$180/n + 180/m + 180/p = 180$

or $1/n + 1/m + 1/p = 1$

There are few solutions

- for the solution: $1/3 + 1/3 + 1/3 = 1$, the three angles of the triangle are $180/3= 60$ degrees, so the triangle is equilateral

- the solution $1/4 + 1/4 + 1/2 = 1$ corresponds to angles of *45, 45* and *90* degrees, hence to a right angled isosceles triangle

- and finally the solution $1/2 + 1/6 + 1/3 = 1$ corresponds to angles of *90, 30 and 60* degrees, hence to half of an equilateral triangle

Thus there are only three triangles!

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